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| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Linear Sys | | Line in xy-plane: ax+by = c, plane: ax+by+cz = d  Linear sys: finite set of linear eqn in variables x1, x2... xn  Solution set: set of all soln to linear sys, {(t,2t-1)|t}  Zero system: all constant are zero,  Inconsistent sys: sys has no soln  Every LS must either have no soln, only 1 soln, or infinitely many sol | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | In general: linear eqn in n vars x1, x2... xn has form a1x1 +... anxn = b, where a1...an and b are constants  General soln: expression that gives all the soln,  Nonzerosys: not zero sys  Consistent sys: ≥ 1 soln | | | | | | | | | | | | |
| ERO | | System of linear eqn  x1 + x2 + 2x3 = 9  2x1 + 4x2 - 3x3 = 1  3x1 + 6x2 - 5x3 = 0 | | | | | | | | Augmented matrix | | | | | | | | | | | | | | | | | | | | 1. Multiply row by nonzero constant, aRi  2. Interchange 2 rows, Ri Rj  3. Add multiple of 1 row to another row, Ri + aRj | | | | | | | | | | | | | | | | | | |
| Row equivalent: If one augmented matrix can be obtained another by series of ERO  If 2 augmented matrix are row-equivalent both have same set of soln | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| REF | | REF: 1. Zero rows are all at bottom of matrix  2. In any 2 successive nonzero rows, leading entry/pivot point in lower row is to the right of leading entry of higher row | | | | | | | | | | | | | | | | | | | | | | | | | RREF: 3. Leading entry of every nonzero row is 1  4. In each pivot col, except pivot pt, all other entries = 0  Back-substitution: finding soln from REF/RREF | | | | | | | | | | | | | | | | | | | | | |
| Pivot col: col containing pivot pt; Non-pivot col: col not containing pivot pt | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Gaussian Elimination | | | | | | | | | Gaussian Elimination: get to REF.  RREF is unique but REF is not unique | | | | | | | | | | | | | | | | | | Gauss-Jordan Elimination: get to RREF | | | | | | | | | | | | | | | | | | | | | |
| LS is inconsistent it last col of REF is pivot col (row with nonzero last entry but 0 elsewhere)  LS has 1 soln if except last col, every col of REF is pivot col  LS has infinitely many soln if except for last col, REF has at least 1 more non-pivot col | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| For m x 3 matrix, REF ≤ 3 nonzero row  3 nonzero row (0 free parameter): intersect at pt  2 nonzero row (1 free parameter): intersect at line | | | | | | | | | | | | | | | | | | 1 nonzero row (2 free parameter): intersect at plane  3 zero row (3 free parameter): whole R3 space | | | | | | | | | | | | | | | | | | | | | |
| Homogene-ous sys | | | | | | | | | Homogeneous sys if Ax = b, where b = 0  Trivial soln: x1 = ... = xn = 0 is always a soln to homogeneous sys  Homogeneous sys has either only trivial soln or infinitely many soln (including trivial soln)  Homogeneous sys with more unknown that eqn has infinitely many soln | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | Non-homogeneous sys if not homogeneous  Always consistent | |
| Matrix | Entries: nums in matrix, (i,j)-entry: num in ith row, jth col of matrix  Size: m x n (num of rows x num of cols)  Col matrix: only 1 column; Row matrix: only 1 row  Sq matrix: same num of rows and cols. Size of sq matrix = order n  Diagonal entries: i = j. Non-diagonal entries: i ≠ j  Diagonal matrix: sq matrix and non-diagonal entries = 0  Scalar matrix: diagonal matrix, diagonal entries have same val | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | Identity matrix, I: diagonal matrix, diagonal entries =1  Zero matrix: all entries = 0  Symmetric: square matrix and aij = aji or A = AT  Upper triangular: square matrix, entries = 0 below diagonal entries  Lower triangular: square matrix, entries = 0 above diagonal entries | | | | | | | | | | | | | | | | |
| Matrix ops | | 2 matrix are equal if they have same size and entries are equal  Ax = b  A: coefficient matrix, x: variable matrix, b: constant matrix   |  |  | | --- | --- | | Matrix Multiplication only when num of cols of A = num of rows of B | | | Let A = (aij)m x p and B = (bij)p x n. (i,j)-entry of AB = "sum (row of A x col of B)" | | | Matrix multiplication not commutative: pre-multiplication of A to B, AB ≠ BA, post-multiplication of A to B | | | 1. Associative Law, A(BC) = (AB)C  2. Distributive Law, A(B1 + B2) = AB1 + AB2 | | | 3. AB = 0 ≠> A = 0 or B = 0 | 4. A0 = 0A = 0 | | 5. c(AB) = (cA)B = A(cB) | 6. AI = IA = A | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  |  | | --- | --- | | Addition and scalar multiplication | | | 1. A ± B = (aij ± bij)m x n | 2. cA = (caij)m x n | | Commutative Law: A+B = B+A | | | Associative Law: A+(B+C) = (A+B)+C | | | 3. c(A+B) = cA + cB | 4. (c+d)A = cA + dA | | 5. c(dA) = (cd)A = d(cA) | 6. A+0 = 0+A = A | | 7. A-A = 0 | 8. 0A = 0 |  |  |  | | --- | --- | | Let A be sq matrix and n a nonngeative int | | | n = 0: An = I | n ≥ 1: An = AA...A n times | | AmAn = Am+n | (AB)n ≠ AnBn |  |  |  | | --- | --- | | Transpose. rows of A = cols of AT (aij = aTji) | | | (AT)T = A | (A+B)T = AT + BT | | (cA)T = cAT | (AB)T = BTAT | | | | | | | | | | | | | | | | | | | |
| Inverses | | | Let A be sq matrix of order n. A is invertible if sq matrix B of order n s.t. AB = I and BA = I. Then B is inverse of A  singular: no inverse  If A is invertible and AB1 = AB2, then B1 = B2 (opp is false)  If A is invertible and C1A = C2A, then C1 = C2  Inverse is unique  Product of invertible matrices will be invertible | | | | | | | | | | | | | | | | | | | | | |  |  | | --- | --- | | (cA)-1 = (1/c)A-1 | (AT)-1 = (A-1)T | | (A-1)-1 = A | (AB)-1 = B-1A-1 | | A-n = (A-1)n = A-1A-1...A-1 n times | An is invertible | | ArAs = Ar+s for any int r,s | (An)-1 = A-n |   For 2 x 2 matrix,= = adj(A)  (A|I) -rref-> (I|A-1) | | | | | | | | | | | | | | | | | | | | | | | | |
| Elementary matrices | | | | | | | cRi, E.g. cR2, EA, E =  Ri Rj, E.g. R2 R3, E =  Ri + cRj, E.g. R3 + 2R1, E = | | | | | | | | | | | | | | | | | (1/c)R2, E-1 =  R2 R3, E-1 = E  R3 - 2R1, E-1 = | | | | | | | | | | | | | | | | | | | | | | | | det(E) = c  det(E) = -1  det(E) = 1 |
| Elementary matrix: sq matrix obtained from **I** by performing a single ERO (all E defined are elementary matrices)  All elementary matrices are invertible and their inverse are also elementary matrices  En...E2E1A = B, then A = E1-1E2-1...En-1B | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Following are equivalent where A is n x n matrix:  1. A is invertible (not singular)  2. A has a left inverse  3. A has a right inverse  4. Ax = 0 has only trivial soln  5. RREF of A is I  6. A is a product of elementary matrices  7. For any b, Ax = b has a unique soln | | | | | | | | | | | | | 8. det(A) ≠ 0  9. Rows/cols of A spans  10. Rows/cols of A are LI (9 & 10 combine to become basis of )  11. rank(A) = n (full rank)  12. nullity(A) = 0  13. 0 is not an eigenvalue of A  14. LT T is injective (Ker(T) = {0})  15. LT T is surjective (R(T) = ) | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Let A,B be sq matrices of same size. If AB = I, then A,B are both invertible  Let A,B be sq matrices of same order. If A is singular, then AB and BA are singular  Post multiplying A by elementary matrix, E: performing elementary column operations (ECO)  Opp of pt 4 true: not invertible = Ax = 0 has infinite soln | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Determi-nants | | | | | | | | | Let A = (aij) be n x n matrix. Let Mij be (n-1) x (n-1) matrix obtained from A by deleting ith row and jth col.  det(A) = , where Aij = (-1)i+jdet(Mij) = (i,j)-cofactor of A ( det(A) = A[adj(A)] )  This mtd of finding det = cofactor expansion. Cofactor expansion can be done along any row/col   |  |  | | --- | --- | | If A is a triangular matrix, det(A) = product of diagonal entries | If sq matrix has 2 same rows/cols, then det = 0 | | det(A) = det(AT) | det(cA) = cndet(A) | | det(AB) = det(A)det(B) | If A is invertible, det(A-1) = 1/det(A) |   So det(A) = det(En)\*...\*det(E2)\*det(E1)\*det(rref(A)) = det(En)\*...\*det(E2)\*det(E1)\*prd of diagonal entries of rref(A)  adj(A) = = , where Aij is the (i,j)-cofactor of A = (-1)i+jdet(Mij)  A[adj(A)] = det(A)I. If A is invertible, then A[adj(A)] = I and A-1 = adj(A) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Cramer's Rule: Ax = b. If A invertible, sys only has 1 soln. Let Ai be matrix obtained from A by replacing ith col of A by b | | | | | | | | | | | | | | | | | | x = | | | | | | | | | | | | | | | | | | | | | |
| Euclidean n-Space | | | | | | n-vector/ordered n-tuple of real numbers has form (u1, u2, ... un), where ui is ith coordinate of n-vector  n-vector can be represented as row or column vector  Euclidean n-space = set of all n-vectors =  Let v be an n-vector v | | | | | | | | | | | | | | | | | Implicit form of subset:  {(u1, u2, u3, u4)| u1 = 0 and u2 = u4} OR {(x,y,z)|x - 2y + z = 1}  Explicit form of subset:  {(0, a, b, a) | a, b } OR {(0,0,1) + s(2,1,0)| s }  |S| = num of elems in S | | | | | | | | | | | | | | | | | | | | | | | | | |
| Linear Combi & Linear Span | | | | | | | | Let S = { u1...uk}. Linear combi of u1...uk = c1u1 + ... + ckuk.  Linear Span of S = {c1u1 + ... + ckuk |c1...ck } = span(S) | | | | | | | | | | | | | | | | | | | rref(u1 u2 ... uk |v). If sys consistent v is LC of u1 ... uk  all LC of u1...uk | | | | | | | | | | | | | | | | | | | | | |
| rref(u1 u2 ... uk). If sys has no zero row sys is consistent regardless of values of x, y, z... and span(S) = | | | | | | | | | | | | | | | | | | | Conversely, if sys has zero row sys not always consistent and span(S) ≠ | | | | | | | | | | | | | | | | | | | | | |
| If k < n S cannot span | | | 0 span(S) | | | | | | | For any v1,...vr span(S) and c1...cr , then c1v1 +... + crvr span(S) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Let S1 = {u1...uk} and S2 = {v1...vm} be subsets of . span(S1) span(S2) iff each ui is LC of v1, ...vm | | | | | | | | | | | | | | | | | | | span(S1) = span(S2) iff  span(S1) span(S2) and span(S2) span(S1) | | | | | | | | | | | | | | | | | | | | | |
| If uk is LC of u1...uk-1, then span{u1...uk-1} = span{u1...uk-1, uk}. uk is a redundant vector | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| |  |  |  |  |  | | --- | --- | --- | --- | --- | |  |  |  |  |  | | span{u} | {cu|c } | line through origin | {(x,y)|u2x-u1y = 0} | {(cu1, cu2, cu3)|c } | | span{u,v}, where u,v not parallel | {su + tv|s,t } | plane containing origin |  | {(x,y,z)|ax+by+cz = 0} | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Subspaces | | | | | | | | | Let V be subset of . V is a subspace of if V = span(S)  OR  V is a subspace of if it contains 0 and closure under addition and scalar multiplication, i.e. u,v V and a,b , au+bv V | | | | | | | | | | | | | | | | | | | | | Let 0 be zero vector in . span{0} is subspace of and aka zero space  is also a subspace of  Soln set of homog sys is a subspace of = soln space  (Rm: mx1 vectors but Rn: nx1) | | | | | | | | | | | | | | | | | | |
| Linear Indepen-dence | | | | | Let S = {u1,...uk} be set of vectors in  S is LI set iff c1u1 + c2u2 +...+ckuk = 0 (Ax = 0) only has trivial soln  If c1u1 +...+ckuk = 0 has non trivial soln, then S is a linear dependent set | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | rref([u1 u2 ... uk]): If no non-pivot col trivial soln LI  If have non-pivot col infinite soln not LI | | | | | | | | |
| S = {u}. If u = 0, then S is linearly dependent  S = {u,v}. If u = cv, then S is linearly dependent | | | | | | | | | | | | | | | | | | | | | | | | | As long as 0 is in a set, set would be linearly dependent  is LI | | | | | | | | | | | | | | | | | | |
| S is linearly dependent iff at least 1 vector ui in S is a LC of other vectors in S, i.e. ui = a1u1 + ...+ ai-1ui-1 + a­i+1ui+1 + ... + akuk | | | | | | | | | | | | | | | | | | | | | | | | | | | | | S is LI iff no vector in S is a LC of other vectors in S  If S is LI, there is no redundant vector in S | | | | | | | | | | | | | | |
| If k > n, then S is linearly dependent | | | | | | | | | | | | | | | | | | | | | | | | | | | | | k unknowns, n eqn, then sys has non-trivial soln | | | | | | | | | | | | | | |
| 2 vectors are linearly dependent if on same line | | | | | | | | | | | | | | | | | | 3 vectors are linearly dependent if on same line/same plane | | | | | | | | | | | | | | | | | | | | | | | | | |
| Let u1,...uk be LI vectors in . If uk+1 is a vector in and not LC of u1,...uk. Then u1,...uk, uk+1 is also LI | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Bases | | | | V is vector space if V = or V is a subspace of  Let W be a vector space. V is also a subspace of W if V is a vector space contained in W | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Let S = {u1,...uk} be a subset of vector space V. S is basis for V if 1. S is LI 2. V = span(S)  S is basis for Rn iff 1. k = n and 2. A is invertible (i.e. RREF of A = I) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Except zero space, any vector space has infinitely many diff bases (not unique) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | Basis for {0} is | | | |
| If S = {u1,...uk} is basis for vector space V, and v is vector in V. Then, there is unique values for ci s.t. v = c1u1 +...+ ckuk  Coefficients ci are the coordinates of v relative to basis S. (v)S = (c1, ...ck). [v]S is col form of (v)S  Standard basis for . e1 = (1,0,...,0), e2 = (0,1,...,0)..., en = (0,...,0,1) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Let S be basis for vector space V | | | | | | | | | For any u,v V, u = v iff (u)S = (v)S  For any v1, v2, ...vr V, (c1v1 +...+ crvr)S = c1(v1)S +...+ cr(vr)S | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Let S be basis for vector space V and |S| = k. Let v1, v2,...vr be vectors in V  1. v1,...vr are linearly dependent/indep iff (v1)S,...(vr)S are linearly dependent/indep vectors in  2. span{v1,...vr} = V iff span{(v1)S,...(vr)S} = | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| If S and T are bases for subspace V, then |S| = |T| | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Dimensions | | | | | | | | | Let V be vector space with basis with k vectors  dim(V) = num of vectors in basis for V (dim({0}) = 0) | | | | | | | | | | | | | | | | | | | | | | | | | | | | 1. Any subset of V with > k vectors is always LD  2. any subset of V with < k vectors cannot span V | | | | | | | | | | | |
| dim = 1 line through origin | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | dim = 2 plane containing origin | | | | | | |
| dim of soln space = num of parameters needed for soln of homogeneous sys = num of non-pivot cols | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| S is basis for V if 1. |S| = dim(V)  2. S is subset of V 3. S is LI | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | S is basis for V if 1. V span(S) (textbook: =)  2. |S| = dim(V) | | | | | | | | | |
| Let U be subspace of vector space V. Then dim(U) ≤ dim(V) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | U = V iff dim(U) = dim(V) | | | | | | |
|  | | | | | | | | | Let V, W be subspace of . | | | | | | dim(V+W) = dim(V) + dim(W) - dim(VW) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Transition Matrices | | | | | | | | Let S = {u1,...uk}, T = {v1,...vk} be 2 bases for vector space V. Let w be a vector in V  [w]T = P[w]S where P = ([u1]T, [u2]T, ..., [uk]T) = transition matrix from S to T | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | rref([v1 v2 ... vk |u1|u2|...|uk]):  P = RHS of rref (exclude zero row) | | |
| Let S, T be 2 bases of vector space, and P be transition matrix from S to T. P must be sq matrix.  Then 1. P is invertible and 2. P-1 is transition matrix from T to S (P-1 = ([v1]S, [v2]S, ..., [vk]S)) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| [u]S = [2v + w]S = 2[v]S + [w]S | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Row space & column space | Let A be m x n matrix  row space of A = span{r1, ...,rm} a subspace of Rn = col space of AT  col space of A = span{c1, ...,cn} a subspace of Rm = row space of AT  Row /col space of 0 = zero space, Row /col space of I = Rn  Let A and B be row equivalent matrices. row space of A = row space of B | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | Nonzero rows of REF of A will form basis for row space of A  Row equivalent matrices preserve linear dependency of cols (i.e. from REF we can tell which columns of A are a LC of other cols)  Cols in A corresponding to pivot cols of REF of A will form basis for col space of A | | | | | | | | | | | | | | | | | |
| Finding basis for linear span (use row space/ col space mtd if need original vectors)  Ax = b has solution b is a LC of cols of A  Col space of A = {Au|u Rn }  Nullspace of A = {u Rn | Au = 0} | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | Extending set to basis (form matrix with vectors in row form, ref and create new vectors with leading entries at non-pivot cols)  Col(AB) Col(A)  Suppose AB = 0. Then Col(B) Null(A) | | | | | | | | | | | | | | | | | |
| Ranks | | | | | | Let A be m x n matrix. rank(A) = dim of row/col space of A  rank(A) ≤ min{m, n}.  Full rank: rank(A) = min{m, n}  Sq matrix A is full rank iff det(A) ≠ 0 iff Col(A) / Row(A) = Rn iff rows/cols of A form basis for Rn | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | rank 0 = 0 matrix  rank(A) = rank(AT) for any matrix A  Ax = b consistent iff rank(A) = rank(A|b)  rank(AB) ≤ min{rank(A), rank(B)}  rank(A + B) ≤ rank(A) + rank(B)  If A invertible, rank(AB) = rank(B) | | | | | | | | |
| Nullspace & Nullities | | | | | | Let A be m x n matrix.  Nullspace of A: soln space of Ax = 0, is subspace of Rn  nullity(A) = dim of nullspace of A, is ≤ n, = num of free params in general soln (non-pivot cols)  Dimension thm: rank(A) + nullity(A) = n | | | | | | | | | | | | | | | | General soln of Ax = b = [(general soln of Ax = 0) + particular soln for Ax = b]  Soln set of Ax = b = {u+v|u nullspace of A}, and v is particular soln for Ax = b  If Ax = b is consistent, sys has only 1 soln iff nullspace of A = {0} | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Solution set of homogeneous linear sys Ax = 0 is always a subspace of Rn, where A is m x n matrix = nullspace of A | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|  | | | | | | Let A be a m x n matrix | | | | | | | | If m > n, A cannot have a right inverse  If n > m, A cannot have a left inverse | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | If rank(A) = n, A has a left inverse  If rank(A) = m, A has a right inverse | | | | |
| Inner/ dot/ scalar product | | | | | | | | | ||u|| = length/norm of u =  cos = (derived from cosine rule)  (uv = u1v1 + u2v2 +... + ukvk) And uu = u12 + u22 + ... + un2  Unit vectors: ||u|| = 1  If u, v are row vectors, uv = uvT  If u, v are col vectors, uv = uTv | | | | | | | | | | | | | | | | | | | | 1. uv = vu (commutative law)  2. (u + v)w = uw + vw. w(u + v) = wu + wv  3. (cu)v = u(cv) = c(uv)  4. ||cu|| = |c| ||u|| (|c| is abs value)  5. uu ≥ 0. uu = 0 iff u = 0  - Av = 0 iff ATAv = 0  6. Cauchy-Schwarz inequality: |uv| ≤ ||u|| ||v|| | | | | | | | | | | | | | | | | | | | |
| Orthogonal/ Orthonormal set | | | | | | | | | 1. 2 vectors u, v are orthogonal if uv = 0 (perpendicular)  2. Set S of vectors is orthogonal is every pairs of vectors in S are orthogonal (i.e., u1u2 = 0, u1u3 = 0, ..., uk-1vk = 0)  3. Set S of vectors is orthonormal if S is orthogonal and every vector in S is a unit vector  Standard basis is orthogonal and orthonormal set | | | | | | | | | | | | | | | | | | | | Orthogornal set: {u1, u2, ... uk}  Orthonormal set: {u1, u2, ...,uk}  If S is orthogonal set of nonzero vectors in vector space, then S is linearly independent  1. Orthogonal basis: S is orthogonal & |S| = dim(V)  2. Orthonormal basis: S is orthonormal & |S| = dim(V) | | | | | | | | | | | | | | | | | | | |
|  | | | | | | | | | |  |  | | --- | --- | | To check if set is orthogonal basis:  Let S be set of nonzero vectors in vector space V | | | (i) S is orthonormal and  (ii) span(S) = V | (i) S is orthonormal and  (ii) |S| = dim V | | | | | | | | | | | | | | | | | | | | | Let S {u1, u2,... uk} be orthogonal basis for V.  For any vector w in V, w = c­1u1 + c2u2 + ... + ckuk  (w)s = (c1 c2 ... ck) = ( ... )  If S is orthonormal basis, ||ui||2 = 1 for all i | | | | | | | | | | | | | | | | | | | |
|  | | | | | | | | | Orthogonal set can contain 0 vector set won't be LI and won't be basis  Let S = {u1, ..., uk}. A = (u1 ... uk). S is orthogonal set iff ATA is a diag matrix  S is orthonormal set iff ATA = Ik | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Find normal to subspace | | | | | | | | | Let V be subspace of Rn. Vector n is orthogonal (normal) to subspace V if u is orthogonal to all vectors in V  V has eqn ax + by + cz = 0 normal vector = n = (a, b, c)  For any vector v (x0, y0, z0) in V, nv = ax0 + by0 + cz0 = 0 | | | | | | | | | | | | | | | | | | | | Let subspace V = span{u1, u2, ... uk} in Rn  1. Let v = (x1, x2, ... xn) = normal  2. Convert v u1 = 0, v uk = 0 into homogeneous sys  3. Solve LS | | | | | | | | | | | | | | | | | | | |
|  | | | | | | | | | Let V be subspace of Rn and w a vector in Rn.  w can be decomposed uniquely as w = wp + wn where wp = projection V and vn  p = the projection of vector w onto subspace V w - p is orthogonal to V (p is unique)   |  |  |  | | --- | --- | --- | |  | 1. S = {u1, u2, ..., uk}: an orthogonal basis for V | 2. T = {v1, v2, ..., vk}: an orthonormal basis for V | | p | u1 + u2 + ...+ uk | (w v1)v1 + (w v2)v2 + ... + (w vk)vk | |  | u1 + u2 + ...+ uk = | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Convert a basis to orthogonal basis | | | | | | | | | Use Gram-Schmidt Process (project vector to subspace)   |  |  |  | | --- | --- | --- | |  | vi | wi | | u1 | v1 = u1 | w1 = v1 | | u2 | v2 = u2 - v1 (orthogonal to v1) | w2 = v2 | | u3 | v3 = u3 - v1 - v2 (orthogonal to v1 and v2) |  | | uk | vk = uk - v1 - v2 - ... - vk-1 | wk = vk | | {u1, u2, ..., uk} basis for vector space V | {v1, v2, ..., vk} is orthogonal basis for V | {w1, w2, ..., wk} orthonormal basis for V | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|  | | | | | | | | | W = span{u1, u2, ... uk}. A = (u1, u2, ..., uk)  v iff v ui = 0 i = 1...k  v iff v Null(AT). = null(AT) / null(A) and W = col(A) / row(A) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | u Row(A) iff u Null(AT)  null(A) = null(ATA)  null(AT) = null(AAT) | | | | | | |
| Best Approxi-mations | | | | | | | | | Let V be subspace in Rn and u Rn  p: projection of u onto V = p is best approximation of u in V  dist(u, p) ≤ dist(u, v) for any v in V | | | | | | | | Suppose Ax = b is inconsistent Ax - b ≠ 0  Least sq soln of Ax = b is a vector u in Rn that minimise ||b-Ax||, i.e. ||b-Au|| ≤ ||b-Av|| v in Rn  iff ATAu = ATb p = Au = A(ATA)-1ATb | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Suppose u is least sq soln (Au = projection p of b onto col space of A)  iff Au = p (always consistent since p lies on col space of A)  Suppose A = (u1 u2 u3) Ax = cu1 + du2 + eu3 (LC of cols of A)  All Ax belongs to col space of A | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | A(least sq soln) = projection  least sq soln may not be unique  u might not be in col space of A | | | | | | |
| u is least sq soln of Ax = b (A = (a1 a2 a3)) iff u is soln to ATAx = ATb  Au is projection of b onto V (V = col space of A)  b - Au is orthogonal to V  b - Au is orthogonal to a1, a2, a3 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | AT(b - Au) = (b - Au) = (dot product)  ATAu = ATb u is soln to Ax = p (p = projection of b onto col space of A) | | | | | | | | |
| Orthogonal Matrices | | | | | | | | | Sq matrix A is orthogonal matrix if A-1 = AT AAT = I or ATA = I  (i.e. all orthogonal matrices are invertible)  Product of 2 orthogonal matrix also orthogonal | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | Let A be sq matrix of order n  1. A is orthogonal matrix  2. Rows of A forms orthonormal basis for Rn  3. Cols of A form orthonormal basis for Rn | | | | | | | | | |
| Transition matrix btw orthonormal bases | | | | | | | | | S = {u1, u2, ..., uk}. T = {v1, v2, ..., vk}. P = ([u1]T [u2]T ... [uk]T). Then [w]T = P[w]S  Suppose S and T are 2 orthonormal bases for a vector space  Then transition matrix P from S to T is orthogonal.  So PT is transition matrix from T to S | | | | | | | | | | | | | | | | | | | | | | | | P =  Q = P-1 = = PT | | | | | | | | | | | | | | | |
| Rotation of xy-coordinates | | | | | | | | | S = {(1, 0), (0, 1)}, T = {u1, u2} = {(cos , sin ), (-sin , cos )}  v = [v]S (since standard basis)  [v]T = PT[v]S where P = , PT = | | | | | | | | | | | | | | | | | | | | | | | | | | | | T = new coordinate sys  [v]T = rotating xy-coordinate anticlockwise by  = rotate vector clockwise by | | | | | | | | | | | |
| Eigenvalues, Eigenvectors, Eigenspace | | | | | | | | | Diagonalizing a sq matrix: A = PDP-1 (D is diag matrix). An = PDnP-1  Let A be sq matrix of order n, x be nonzero col vector in Rn  If Ax = x for some scalar , x is an eigenvector of A  is eigenvalue of A associated with eigenvector x  A = (x1 x2)(x1 x2)-1 (xi is eigenvector associated with eigenvalue i) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | If A is a triangular matrix, eigenvalues of A = diag entries  characteristic polynomial of A =  det(I - A)  Let A = PDP-1. Then An = PDnP-1 | | | | | |
| Finding eigen-values | | | | Let A be sq matrix of order n, is eigenvalue of A  Ax = x, for some nonzero col vector x x - Ax = 0 (I - A)x = 0 has non-trivial soln (sys always consistent since homog sys) det(I - A) = 0 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | is an eigenvalue of A det(I - A) = 0  is a root of the characteristic polynomial | | | | | | | |
| Finding eigenvectors | | | | | | | | | det(A) ≠ 0 0 is not an eigenvalue of A  Proof. 0 is not an eigenvalue of A  0 not a root of char polynomial (det(I - A) ≠ 0)  det(0I - A) ≠ 0 det(-A) ≠ 0 (-1)ndet(A) ≠ 0 det(A) ≠ 0 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | Finding eigenvectors.  Ax = x, for some nonzero col vector x  x - Ax = 0 (I - A)x = 0  Then just solve this homogeneous sys to find x | | | | | | | | | | |
| Eigenspace | | | | | | | | | = eigenspace of A associated with eigenvalue = soln space of LS (I - A)x = 0 (has nontrivial soln)  If u is a nonzero vector in , then u is an eigenvector in A associated with eigenvalue | | | | | | | | | | | | | | | | | | | | | | | | | | Just find general soln of (I - A)x = 0, then eigenspace is span by the vector  Although 0 in eigenspace, 0 cannot be eigenvector as eigenvector always nonzero vector | | | | | | | | | | | | | |
| Diagonal-ization | | | | | | | | | A square matrix A is diagonalizable if an invertible matrix P s.t. P-1AP is a diagonal matrix, i.e.  A = PDP-1 or P-1AP = D  Matrix P diagonalizes A | | | | | | | | | | | | | | | | | | | Let A be sq matrix of order n.  A is diagonalizable A has n LI eigenvectors.  A has n distinct eigenvalues () A is diagonalizable  Diagonal matrices are diagonalizable | | | | | | | | | | | | | | | | | | | | |
|  | | | | | | | | | Note that BD = (b1 b2 ... bn)D = (d1b1 d2b2 ... dnbn) if D is a diagonal matrix with diagonal entries d1 d2... dn | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Check if A is diagonali-zable | | | | | | | | | 1. Solve det(I - A) = 0 to find all eigenvalues  2. For each eigenvalues, find basis for eigenspace  3. Let S = ... . (S is always LI)  a) If |S| < n, A is not digonalizable  b) If |S| = n, A is diagonalizable | | | | | | | | | | | | | | | | | | | det(I - A) = ..., where ri = multiplicity, then dim() ≤ ri  A is diagonalizable iff dim() = ri for all  A only has 1 eigenvalue and is a scalar matrix A is diagonalizable | | | | | | | | | | | | | | | | | | | | |
|  | | | | | | | | | In general, a0 = s, a1 = t, an = pan-1 + qan-1 . Then recurrence matrix A = , = An = PDnP-1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Orthogonal Diagonali-zation | | | | | | | A sq matrix A is orthogonally diagonalizable if an orthogonal matrix P s.t. PTAP is a diagonal matrix | | | | | | | | | | | | | | | | | | Matrix P orthogonally diagonalizes A  Sq matrix is orthogonally diagonalizable iff it is symmetric | | | | | | | | | | | | | | | | | | | | | | | |
| 1. Solve det(I - A) = 0 to find all eigenvalues  2. For each , a) find basis for eigenspace  b) Gram-Schmidt to transform into orthonormal basis  3. Let T = ... . T = {v1, v2, ..., vn}. (T is orthonormal)  Then P = (v1 v2 ... vn) is orthogonal matrix that diagonalizes A | | | | | | | | | | | | | | | | | | | | | | | | Eigenvalues of symmetric matrix are always real nums  Let A be a symmetrix matrix, and det(I - A) = ..., then  dim() = ri, i.e. A is always diagonalizable  r1 + r2 + ... + rk = order of A  dim+ dim+...+ dim = num of LI eigenvectors | | | | | | | | | | | | | | | | | |
|  | | | | | | | If A is invertible and diagonalizable, then A-1 also diagonalizable  If A diagonalizable, A-1 also diagonalizable | | | | | | | | | | | | | | | | | | | | | | | | If A and B are orthogonally diagonalizable, then A+B also orthogonally diagonalizable (since sum of symmetric matrix still symmetric) | | | | | | | | | | | | | | | | | |
| Linear Transform-ation | | | | | | | T = (formula)  T: Rn Rm is LT iff T(u) = Au u in Rn  T is a linear transformation from Rn to Rm. A is the standard matrix of the linear transformation, A = m x n  Rn: domain of T, Rm: codomain of T | | | | | | | | | | | | | | | | | | I: Rn Rn: the identity transformation, i.e. I(u) = u, A = I  O: Rn Rm: the zero transformation, i.e. O(u) = 0, A = 0m x n  If T: Rn Rm is a linear transformation, then  1. T(0) = 0, i.e. A0 = 0  2. T(c1u1 + c2u2 + ... + ckuk) = c1T(u1) + c2T(u2) +...+ ckT(uk)  OR T(au + bv) = aT(u) + bT(v) | | | | | | | | | | | | | | | | | | | | | | | |
|  | | | | | | | If linear transformation T: Rn Rn, i.e. domain = codomain, then T is a linear operator on Rn, and standard matrix for T is a sq matrix | | | | | | | | | | | | If given T(u1) = v1, T(u2) = v2, T(u3) = v3  Can find image of any other vector if u1, u2, u3 form basis for R3.  Find LC of u1 u2 u3 = u4. Then T(u4) = LC of v1 v2 v3 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Finding A | | | | | | | Given T(u1) = v1, T(u2) = v2, T(u3) = v3, to find formula for T,  1. Direct Gaussian elimination  = c1u1 + c2u2 + c3u3, to find ci  Then T = c1v1 + c2v2 + c3v3 | | | | | | | | | | | | | | | | | | | | | 2. Find T(e1), T(e2) , T(e3)  Note T(ei) = Aei = ith col of A. So A = (T(e1) T(e2) ... T(en))  Find e1, e2, e3 in terms of u1, u2, u3  Then T(e1) = c1v1 + c2v2 + c3v3 | | | | | | | | | | | | | | | | | | | | |
| Finding A & Composite | | | | | | | 3. Stack matrices  So A =  Then A = | | | | | | | | | Let S: Rn Rm and T: Rm Rk be LT. Then (T S)(u) = T(S(u)) u in Rn  (T S) = T to find final formula OR (T S) = BA | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Range | T: Rn Rmis linear transformation. Range = possible images  Range of T = R(T) = set of images of T = {T(u) |u Rn} (explicit set notation) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | R(T) Rm  R(T) codomain of T | | | | | |
|  | R(T) = = span  explicit set notation linear span form  A = ,R(T) = col space of A. R(T) is subspace of Rm  rank(T) = dim of R(T) = dim of col space of A = rank(A) | | | | | | | | | | | | | | | | | | | | Finding basis for range of T = finding basis for col space of A  1. If formula of T:Rn Rm is given  R(T) = {formula in x1, x2, ..., xn|x1, x2, ..., xn R}  2. If standard matrix A given,  R(T) = span{cols of A} = span{T(e1), T(e2), ..., T(en)}  3. If image of basis {u1, u2, ..., un} for Rn  R(T) = span{T(u1), T(u2), ..., T(un)}. Find basis from this set | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Kernel | | | | | | | Let T: Rn Rm. The kernel of T = ker(T) = set of vectors in Rn whose image is zero vector in Rm = { u Rn| T(u) = 0}  ker(T) Rn | | | | | | | | | | | | | | | | | | | ker(T) = all u s.t T(u) = 0 = all u s.t. Au = 0 = soln space of Ax = 0 = nullspace of A, ker(T) is subapce of Rn | | | | | | | | | | | | | | | | | | | | | | |
|  | | | | | | | dim of ker(T) = nullity(T) = nullity(A)  rank(T) + nullity(T) = rank(A) + nullity(A) = n | | | | | Proving qns. Let T: Rn Rm be linear transformation   |  |  |  |  | | --- | --- | --- | --- | |  | ker(T) = { u Rn| T(u) = 0} | R(T) = {T(u) |u Rn} |  | | Given | v ker(T) | v R(T) | WTS | | Follow up with | T(u) = 0 | v = T(u) for some u Rn | try to show | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|  | | | | | | | T: Rn Rm. T is injective if whenever T(u) = T(v), then u = v. iff Ker(T) = {0} iff nullity(T) = 0  T is surjective if for any w Rm, there is a u Rn s.t. T(u) = w. iff R(T) = Rm iff rank(T) = m  If n = m, T is injective iff T is surjective | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |

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